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HW 9

Solutions

1 (a)

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So, $T(\vec{v}) = A\vec{v}$ where $A = \begin{pmatrix} 1 & 1 \\ 2 & -4 \end{pmatrix}$.

Thus, T is a linear transformation

Note: $A = [T]_\gamma$ where $\gamma = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$ is the standard basis

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1 (b)

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ -2(1)+4(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2 \\ -2(1)+4(2) \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus,

$$[T]_\beta = \left(\left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_\beta \middle| \left[T\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_\beta \right) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

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1 (c)

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

So, $[\vec{v}]_\beta = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Also,

$$T(\vec{v}) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix} - 3\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\-2\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}a+b\\a+2b\end{pmatrix}$$

$$a+b=1$$
$$a+2b=-2$$
$$a=4,\ b=-3$$

So,

$$[T(\vec{v})]_\beta = \begin{pmatrix}4\\-3\end{pmatrix}$$

And

$$[T]_\beta[\vec{v}]_\beta = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}$$

$$= \begin{pmatrix}2\cdot 2 + 0(-1)\\0\cdot 2 + 3(-1)\end{pmatrix} = \begin{pmatrix}4\\-3\end{pmatrix} = [T(\vec{v})]_\beta$$

1(d)

$$\vec{v} = \begin{pmatrix}2\\1\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} - 1\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}2\\1\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}a+b\\a+2b\end{pmatrix}$$

$$\begin{matrix}2=a+b\\1=a+2b\end{matrix} \rightarrow \begin{matrix}a=3\\b=-1\end{matrix}$$

So, $[\vec{v}]_\beta = \begin{pmatrix}3\\-1\end{pmatrix}$

And,

$$T(\vec{v}) = T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2+1\\-2\cdot 2+4\cdot 1\end{pmatrix}$$

$$= \begin{pmatrix}3\\0\end{pmatrix} = 6\begin{pmatrix}1\\1\end{pmatrix} - 3\begin{pmatrix}1\\2\end{pmatrix}$$

So, $[T(\vec{v})]_\beta = \begin{pmatrix}6\\-3\end{pmatrix}$

And,

$$[T]_\beta [\vec{v}]_\beta = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 0(-1) \\ 0 \cdot 3 + 3(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -3 \end{pmatrix} = [T(\vec{v})]_\beta$$

② (a)

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So, $T(\vec{v}) = A\vec{v}$ where $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Thus, T is a linear transformation

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② (b)

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1-0 \\ -1+0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -1\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

I did these steps in my head since it was easy to solve

So,

$$[T]_\beta = \left(\left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_\beta \middle| \left[T\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]_\beta \right) = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$$

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② (c)

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

So, $[\vec{v}]_\beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

And $T(\vec{v}) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

So, $[T(\vec{v})]_\beta = \begin{pmatrix}1\\0\end{pmatrix}$

Thus,

$$[T]_\beta[\vec{v}]_\beta = \begin{pmatrix}2 & -1\\2 & 0\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}0+1\\0+0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = [T(\vec{v})]_\beta$$

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② (d)

$\vec{v} = \begin{pmatrix}0\\1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\end{pmatrix} + 1\cdot\begin{pmatrix}-1\\0\end{pmatrix}$

So, $[\vec{v}]_\beta = \begin{pmatrix}1\\1\end{pmatrix}$

And $T(\vec{v}) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0-1\\0+1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\end{pmatrix} + 2\cdot\begin{pmatrix}-1\\0\end{pmatrix}$

So, $[T(\vec{v})]_\beta = \begin{pmatrix}1\\2\end{pmatrix}$

Thus,

$$[T]_\beta[\vec{v}]_\beta = \begin{pmatrix}2 & -1\\2 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2-1\\2+0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} = [T(\vec{v})]_\beta$$

3 (a)

We have

$$T(\vec{a}) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\3\\0\end{pmatrix} = \underbrace{3\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}-1\\0\\1\end{pmatrix} + 0\begin{pmatrix}1\\2\\1\end{pmatrix}}_{3\vec{a} + 0\vec{b} + 0\vec{c}}$$

$$T(\vec{b}) = T\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\0\\3\end{pmatrix} = \underbrace{0\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}-1\\0\\1\end{pmatrix} + 0\begin{pmatrix}1\\2\\1\end{pmatrix}}_{0\vec{a} + 3\vec{b} + 0\vec{c}}$$

$$T(\vec{c}) = T\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}5\\10\\5\end{pmatrix} = \underbrace{0\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}-1\\0\\1\end{pmatrix} + 5\begin{pmatrix}1\\2\\1\end{pmatrix}}_{0\vec{a} + 0\vec{b} + 5\vec{c}}$$

Thus,

$$[T]_\beta = \left([T(\vec{a})]_\beta \;\middle|\; [T(\vec{b})]_\beta \;\middle|\; [T(\vec{c})]_\beta \right)$$

$$= \begin{pmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 5\end{pmatrix}$$

③ (b)

Since $\vec{v} = 1\cdot\vec{a} + 1\cdot\vec{b} + 1\cdot\vec{c}$ we have $[\vec{v}]_\beta = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Then

$$[T]_\beta [\vec{v}]_\beta = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$$

And

$$T(\vec{v}) = T\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4\cdot 0 + 2 \\ 2\cdot 0 + 3\cdot 3 + 2\cdot 2 \\ 2\cdot 0 + 4\cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 13 \\ 8 \end{pmatrix}$$

Note that

$$3\cdot\vec{a} + 3\cdot\vec{b} + 5\cdot\vec{c} = 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 13 \\ 8 \end{pmatrix} = T(\vec{v})$$

Thus, $[T(\vec{v})]_\beta = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$

So, $[T]_\beta [\vec{v}]_\beta = [T(\vec{v})]_\beta$.